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On Gibbs Measures of Models with Competing Ternary and Binary Interactions and Corresponding von Neumann Algebras II

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In the present paper the Ising model with competing binary (J) and binary (J_1) interactions with spin values ± 1, on a Cayley tree of order 2 is considered. The structure of Gibbs measures for the model is studied. We completely describe the set of all periodic Gibbs measures for the model with respect to any normal subgroup of finite index of a group representation of the Cayley tree. Types of von Neumann algebras, generated by GNS-representation associated with diagonal states corresponding to the translation invariant Gibbs measures, are determined. It is proved that the factors associated with minimal and maximal Gibbs states are isomorphic, and if they are of type III_λ then the factor associated with the unordered phase of the model can be considered as a subfactors of these factors respectively. Some concrete examples of factors are given too.

KEY WORDS: Cayley tree; Ising model; competing interactions; Gibbs measure; GNS-construction; Hamiltonian; von Neumann algebra.

1. INTRODUCTION

The present paper is a continuation of the paper.[(11)] In that paper it has been given motivations to study of the Ising models with competing interactions and we have investigated the Ising model with ternary interactions on a Cayley tree.

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Recall that the Cayley tree Γ^k of order $k \geqslant 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k+1$ edges. Let $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. The vertices x and y are called *nearest neighbors*, if there exists an edge connecting them, such vertices are denoted by $\langle x, y\rangle$. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

We set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \cup_{m=1}^n W_m,$$

for an arbitrary point $x^0 \in V$.

Denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

this set is called a set of direct successors of x.

Two vertices $x, y \in V$ is called *one level next-nearest-neighbor vertices* if there is a vertex $z \in V$ such that $x, y \in S(z)$, and they are denoted by $>x, y<$. In this case the vertices x, z, y was called *ternary* and denoted by $\langle x, z, y\rangle$.

In this paper we consider the Ising model with competing interactions, where the spin takes values in the set $\Phi = \{-1, 1\}$, on the Cayley tree which is defined by the following Hamiltonian

$$H(\sigma) = -J \sum_{>x,y<} \sigma(x)\sigma(y) - J_1 \sum_{\langle x,y\rangle} \sigma(x)\sigma(y), \tag{1.1}$$

where $J, J_1 \in \mathbb{R}$ are coupling constants and σ a configuration on V, i.e., $\sigma \in \Omega = \Phi^V$.

The other parts of the paper is organized as follows. In Section 2 using the similar argument as[11] we reduce the problem of describing limit Gibbs measures to the problem of solving a nonlinear functional equation. By means of the obtained equation we construct periodic Gibbs measures and find ground states the model. In Section 3, we determine the types of von Neumann algebras generated by GNS-representation associated with diagonal states corresponding to the translation invariant measures. In addition, we will demonstrate more concrete examples of factors. In Section 4 we discuss the results.

2. ON THE SET OF GIBBS MEASURES

In this section recall the construction of a special class of limiting Gibbs measures for the Ising model on a Cayley tree with competing interactions.

Let $h : x \to \mathbb{R}$ be a real valued function of $x \in V$. Given $n = 1, 2, \ldots$ consider the probability measure $\mu^{(n)}$ on Φ^{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\}, \tag{2.1}$$

Here, $\beta = \frac{1}{T}$ and $\sigma_n : x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp\{-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\},$$

$$H(\sigma_n) = -J \sum_{>x,y<:x,y \in V_n} \sigma_n(x)\sigma_n(y) - J_1 \sum_{\langle x,y\rangle : x,y \in V_n} \sigma_n(x)\sigma_n(y). \tag{2.2}$$

Recall that the consistency condition for $\mu^{(n)}(\sigma_n), n \geqslant 1$ is

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}), \tag{2.3}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

The following statement describes conditions on h_x guaranteeing the consistency condition of measures $\mu^{(n)}(\sigma_n)$. In the sequel for the simplicity we will consider the case $k = 2$.

Theorem 2.1. The measures $\mu^{(n)}(\sigma_n),\ n = 1, 2, \ldots$ satisfy the consistency condition (2.3) if and only if for any $x \in V$ the following equation holds:

$$h_x = \frac{1}{2} \log\left(\frac{\theta_1^2 \theta e^{2(h_y + h_z)} + \theta_1(e^{2h_y} + e^{2h_z}) + \theta}{\theta e^{2(h_y + h_z)} + \theta_1(e^{2h_y} + e^{2h_z}) + \theta_1^2 \theta}\right), \tag{2.4}$$

here $\theta = e^{2\beta J}, \quad \theta_1 = e^{2\beta J_1}$ and $< y, x, z >$ are ternary neighbors.

Proof. Using (2.1) it is easy to see that (2.3) and (2.4) are equivalent. (cf. ref. 11). ∎

This theorem reduces the problem of describing of Gibbs measures to the description of solutions of the functional Eq. (2.4).

According to Proposition 2.1[11] that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geqslant 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$.

Recall that $h = \{h_x : x \in G_k\}$ is $\hat{G}_k$*-periodic* if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in \hat{G}_k$, here $\hat{G}_k$ is a normal subgroup of G_k with finite index. A Gibbs measure is called $\hat{G}_k$*-periodic* if it corresponds to $\hat{G}_k$-periodic function h. If it is G_k-periodic, then this measure is *translation-invariant*.

As before in the sequel we will consider the group G_2.

As in ref. 11 we firstly find translation-invariant solutions of (2.4). This case recently has been investigated in ref. 6. For the sake of completeness and since throughout the paper we will use this result, we recall it.

In this setting (2.4) has the form

$$u = \frac{\theta_1^2 \theta u^2 + 2\theta_1 u + \theta}{\theta u^2 + 2\theta_1 u + \theta_1^2 \theta}, \tag{2.5}$$

where $u = e^{2h}$.

Proposition 2.2 (ref. 6.) If $\theta_1 > \sqrt{3}$ and $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ then for all pairs (θ, θ_1) the equation (2.5) has three positive solutions $u_1^* < u_2^* < u_3^*$, here $u_2^* = 1$. Otherwise (2.5) has a unique solution $u_* = 1$.

Remark 2.1. The numbers u_1^* and u_3^* are the solutions of the following equation

$$u^2 + (1+\alpha)u + 1 = 0, \tag{2.6}$$

here $\alpha = \frac{2\theta_1}{\theta} - \theta_1^2$. Hence, if $\beta \to \infty$ then $u_3^* \to \infty$ and $u_1^* \to 0$.

By μ_1, μ_2, μ_3 we denote Gibbs measures corresponding to these solutions.

Using the same argument as[11] one can prove the following.

Theorem 2.3. For the model (1.1) with parameters $J_1 > 0$ and $J \in \mathbb{R}$ on the Cayley tree Γ^2 the following assertions hold

(i) if $\theta_1 > \sqrt{3}, \theta > \frac{2\theta_1}{\theta_1^2 - 3}$ then the measures μ_1 and μ_3 are extreme;

(ii) in the opposite case there is a Gibbs measure $\mu_*(=\mu_2)$ and it is extreme.

Remark 2.2. This theorem specifies the result obtained in ref. 6, as there it was proved that a phase transition occurs if the above indicated conditions is satisfied, and the extremity was open. The formulated theorem answers that the found Gibbs measures are extreme. In spite of this, further we will show that a phase transition can be occur when the condition of theorem is not satisfied.

Remark 2.3. The measure μ_2 corresponding to the solution $h_x = 0, x \in V$ is the unordered phase, i.e., the spin $\sigma(x)$ takes its values ± 1 with respect to μ_2 with probability $1/2$.

Now we turn to the constructions of periodic Gibbs measures. Let H_0 be a subgroup of index r in G_2, and let $G_2|H_0 = \{H_0, H_1, \ldots, H_{r-1}\}$ be the quotient group. Let $q_i(x) = |S^*(x) \cap H_i|, i = 0, 1, \ldots, r-1;\ N(x) = |\{j : q_j(x) \neq 0\}|$, where $S^*(x)$ is the set of all nearest neighbors of $x \in G_2$. Denote $Q(x) = (q_0(x), q_1(x), \ldots, q_{r-1})$. We note (see ref. 13) that for every $x \in G_2$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e is the identity of G_2) such that

$$\pi_x Q(e) = Q(x). \tag{2.7}$$

It follows from (2.7) that $N(x) = N(e)$ for all $x \in G_2$.

It is clear that each H_0-periodic function h_x is given by

$$\{h_x = h_i \quad \text{for} \quad x \in H_i, i = 0, 1, \ldots, r-1\}.$$

By $G_2^{(2)}$ we denoted in[11] the subgroup of G_2 consisting of all words with even length. This $G_2^{(2)}$ has an index 2.

Theorem 2.4. Let H_0 be a subgroup of finite index in G_2. Then each H_0-periodic Gibbs measure for (1.1) model is either translation-invariant or $G_2^{(2)}-$ periodic.

Proof. Let $f(x, y)$ be function defined as follows

$$f(x, y) = \frac{\theta_1^2 \theta xy + \theta_1(x+y) + \theta}{\theta xy + \theta_1(x+y) + \theta_1^2 \theta}, \quad x, y > 0. \tag{2.8}$$

For $\theta_1 \neq 1$ it is easy to see that $f(u_1, v) = f(u_2, v)$ if and only if $u_1 = u_2$. Also $f(u, v_1) = f(u, v_1)$ if and only if $v_1 = v_2$. Using this property of

$f(u,v)$, by Theorem 2.1 and (2.7) we have

$$h_x = h_y = h_1, \quad \text{if} \quad x, y \in S^*(z), \quad z \in G_2^{(2)};$$
$$h_x = h_y = h_2, \quad \text{if} \quad x, y \in S^*(z), \quad z \in G_2 \setminus G_2^{(2)}.$$

Thus the measures are translational-invariant (if $h_1 = h_2$) or $G_2^{(2)}$-periodic (if $h_1 \neq h_2$). The theorem is proved.

If H_0 is a subgroup of finite index in G_2, then it natural to ask: what condition on H_0 guarantees that each H_0-periodic Gibbs measure to be translation-invariant? We put $I(H_0) = H_0 \cap \{a_1, a_2, a_3\}$, where $a_i, i = 1, 2, 3$ are generators of G_2. ∎

Theorem 2.5. If $I(H_0) \neq \emptyset$, then each H_0-periodic Gibbs measure for (1.1) model is translation-invariant.

Proof. Take $x \in H_0$. We recall that the inclusion $xa_i \in H_0$ holds if and only if $a_i \in H_0$. Since $I(H_0) \neq \emptyset$, there is an element $a_i \in H_0$. Therefore H_0 contains the subset $H_0 a_i = \{xa_i : x \in H_0\}$. By Theorem 2.4 we have $h_x = h_1$ and $h_{xa_i} = h_2$. Since x and xa_i belong to H_0, it follows that $h_x = h_{xa_i} = h_1 = h_2$. Thus each H_0-periodic Gibbs measure is translation-invariant. This proves the theorem. ∎

Theorems 2.4, 2.5 reduce the problem of describing H_0-periodic Gibbs measures with $I(H_0) \neq \emptyset$ to describing the fixed points of the function $f(u,u)$ (see (2.8)) which describe the translation-invariant Gibbs measures. If $I(H_0) = \emptyset$, this problem is reduced to the describing solutions of the system:

$$\begin{cases} u = \dfrac{\theta_1^2 \theta v^2 + 2\theta_1 v + \theta}{\theta v^2 + 2\theta_1 v + \theta_1^2 \theta}, \\ v = \dfrac{\theta_1^2 \theta u^2 + 2\theta_1 u + \theta}{\theta u^2 + 2\theta_1 u + \theta_1^2 \theta}, \end{cases} \tag{2.9}$$

where $u = \exp\{2h_1\}$, $v = \exp\{2h_2\}$.

The analysis of the Eq. (2.9) is carried in the following.

Proposition 2.6. The equation (2.9) has three positive solutions (1,1), (u_*, v_*) and (v_*, u_*) (here $u_* < v_*$) if and only if $\theta_1 < 1/\sqrt{3}$ and $\theta > \dfrac{2\theta_1}{1 - 3\theta_1^2}$. Here u_*, v_* are the solutions of the equation:

$$\theta_1^2 \theta (\theta_1^2 \theta + 2\theta_1 + \theta)(x^2 + 1) + ((\theta_1^2 \theta)^2 + 4\theta_1^3 \theta + 4\theta_1^2 - \theta^2) x = 0. \tag{2.10}$$

Proof. It is clear that (1.1) is a solution of (2.9). The Eq. (2.9) can be written as $u=g(g(u))$, here $g(u)=f(u,u)$. Hence, the solutions of the equations $u=g(u)$ are the solution of (2.9), but they describe only the translation-invariant Gibbs measures. Now we should find solutions of $\frac{g(g(u))-u}{g(u)-u}=0$. After some calculations it can be shown that the last equation has the form (2.10).

Full analysis of the Eq. (2.10) shows that parameters θ, θ_1 must satisfy the condition of the proposition. This completes the proof. ▮

Thus we can formulate the following

Theorem 2.7. For the model (1.1) with respect to any subgroup H_0 of finite index the following assertions hold:

(i) Let $\theta_1>\sqrt{3}$, $\theta>\dfrac{2\theta_1}{\theta_1^2-3}$ be satisfied, then H_0-periodic Gibbs measures coincide with the translation-invariant Gibbs measures.

(ii) Let $\theta_1<1/\sqrt{3}, \theta>\dfrac{2\theta_1}{1-3\theta_1^2}$ be satisfied.

  (a) If $I(H_0)\neq\emptyset$ then H_0-periodic Gibbs measures coincide with the translation-invariant Gibbs measures.

  (b) If $I(H_0)=\emptyset$ then there are three H_0-periodic ($=G_2^{(2)}$-periodic) Gibbs measures μ_{12}, μ_{21} and μ_*. Here the measure μ_* corresponds to the unique solution of Eq (2.5).

Proof. Let the condition (i) be satisfied. Then Proposition 2.6 implies that there is no $G_2^{(2)}$-periodic Gibbs measures in this setting. Therefore, according to Theorem 2.4 we conclude that the H_0-periodic Gibbs measures are translation-invariant. Now let (ii) hold. Then the assertions (a) and (b) immediately follow from Proposition 2.6, Theorems 2.4. and 2.5. This completes the proof. ▮

Remark 2.4. In ref. 11 we have investigated only $G_2^{(2)}$-periodic Gibbs measures, the proved Theorem 2.7 completely describes all periodic Gibbs measures, associated with subgroups of G_2 with finite index, of the model.

Now comparing Theorems 2.3 and 2.7 we infer the following

Corollary 2.8. If $1/\sqrt{3}< \theta_1<\sqrt{3}$ then for the model (1.1) there is no phase transition.

By using the similar argument as in[11] we can prove the extremity of μ_{12}, μ_{21}.

Theorem 2.9. Let $\theta_1<1/\sqrt{3}$ and $\theta>\dfrac{2\theta_1}{1-3\theta_1^2}$ be satisfied. Then the measures μ_{12}, μ_{21} and μ_* are extreme.

Using the analogical way as in ref. 1 and 7 with the aid of measures μ_1, μ_2 and μ_3 one can construct uncountable number of extreme Gibbs measures.

From the construction of the Gibbs measures we easily see that the measures μ_1 and μ_3 depend on parameter β. Now we are interested on the behaviour of these measures when β goes to ∞.

Put

$$\sigma_+ = \{\sigma(x) : \sigma(x) = 1, \ \forall x \in \Gamma^2\},$$
$$\sigma_- = \{\sigma(x) : \sigma(x) = -1, \ \forall x \in \Gamma^2\}.$$

Theorem 2.10. Let $\theta_1 > \sqrt{3}$ and $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$, then

$$\mu_1 \to \delta_{\sigma_-}, \qquad \mu_3 \to \delta_{\sigma_+} \quad \text{as} \quad \beta \to \infty,$$

here δ_σ is a delta-measure concentrated on σ.

Proof. Consider the measure μ_3. This measure corresponds to the function $h_x = h_3$, $x \in V$, here $h_3 > 0$ (see Proposition 2.2). Let us first consider a case:

$$\mu_3(\sigma(x) = 1) = \frac{e^{h_3}}{e^{h_3} + e^{-h_3}} = \frac{u_3^*}{u_3^* + 1} \to 1 \quad \text{as } \beta \to \infty,$$

since $u_3^* \to \infty$ as $\beta \to \infty$, here $x \in V$. Let us turn to the general case. From the condition imposed in the Theorem we find that $J_1 > 0$. Now separately consider two cases.

First case. Let $J > 0$. Then from the form of Hamiltonian (1.1) it is easy to check that $H(\sigma_n|_{V_n}) \geqslant H(\sigma_+|_{V_n})$ for all $\sigma \in \Omega$ and $n > 0$. It follows that

$$\begin{aligned}
\mu_3(\sigma_+|_{V_n}) &= \frac{\exp\{-\beta H(\sigma_+|_{V_n}) + h_3|W_n|\}}{\sum\limits_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp\{-\beta H(\tilde{\sigma}_n) + h_3 \sum\limits_{x \in W_n} \tilde{\sigma}(x)\}} \\
&= \frac{1}{1 + \sum\limits_{\tilde{\sigma}_n \in \Omega_{V_n}, \tilde{\sigma}_n \neq \sigma_+|_{V_n}} \frac{\exp\{-\beta H(\tilde{\sigma}_n) + h_3 \sum\limits_{x \in W_n} \tilde{\sigma}(x)\}}{\exp\{-\beta H(\sigma_+|_{V_n}) + h_3|W_n|\}}} \\
&\geqslant \frac{1}{1 + 1/u_3^*} \to 1 \quad \text{as } \beta \to \infty.
\end{aligned}$$

The last inequality yields that $\mu_3 \to \delta_{\sigma_+}$.

Second case. Let $J<0$. Let us introduce some notations.

$$A(\sigma_n)=\sum_{>x,y<:x,y\in V_n}\sigma(x)\sigma(y),\quad A=A(\sigma_+|_{V_n}),$$
$$B(\sigma_n)=\sum_{\langle x,y\rangle:x,y\in V_n}\sigma(x)\sigma(y),\quad B=B(\sigma_+|_{V_n}),$$
$$C(\sigma_n)=\sum_{x\in V_n}\sigma(x),\quad C=C(\sigma_+|_{V_n}).$$

Then it is easy to see that the following equality holds

$$\mu_3(\sigma_+|_{V_n})=\frac{1}{1+\sum\limits_{\tilde\sigma_n\in\Omega_{V_n},\tilde\sigma_n\neq\sigma_+|_{V_n}}\frac{1}{e^{J\beta(A-A(\tilde\sigma_n))}e^{J_1\beta(B-B(\tilde\sigma_n))}e^{h_3(C-C(\tilde\sigma_n))}}}.$$

We want to show that

$$\sum_{\tilde\sigma_n\in\Omega_{V_n},\tilde\sigma_n\neq\sigma_+|_{V_n}}\frac{1}{e^{J\beta(A-A(\tilde\sigma_n))}e^{J_1\beta(B-B(\tilde\sigma_n))}e^{h_3(C-C(\tilde\sigma_n))}}\to 0 \text{ as } \beta\to\infty.$$

It is enough to prove that

$$\frac{1}{e^{J\beta(A-A(\tilde\sigma_n))}e^{J_1\beta(B-B(\tilde\sigma_n))}e^{h_3(C-C(\tilde\sigma_n))}}\to 0 \quad\text{as}\quad \beta\to\infty$$

for all $\tilde\sigma_n\in\Omega_{V_n}, \tilde\sigma_n\neq\sigma_+|_{V_n}$. We rewrite the last sentence as follows

$$\begin{aligned}\frac{1}{e^{J\beta(A-A(\tilde\sigma_n))}e^{J_1\beta(B-B(\tilde\sigma_n))}e^{h_3(C-C(\tilde\sigma_n))}} &= \frac{1}{\theta^{(A-A(\tilde\sigma_n))/2}\theta_1^{(B-B(\tilde\sigma_n))/2}(u_3^*)^{(C-C(\tilde\sigma_n))/2}}\leqslant\\ &\leqslant\frac{(\theta_1^2-3)^{(A-A(\tilde\sigma_n))/2}}{\theta_1^{(A-A(\tilde\sigma_n)+B-B(\tilde\sigma_n))/2}(u_3^*)^{(C-C(\tilde\sigma_n))/2}}\leqslant\\ &\leqslant\frac{(\theta_1^2-3)^{(A-A(\tilde\sigma_n))/2}}{\theta_1^{(A-A(\tilde\sigma_n)+B-B(\tilde\sigma_n))/2}u_3^*}.\end{aligned}\tag{2.11}$$

here we have used the inequality $\theta>\dfrac{2\theta_1}{\theta_1^2-3}$.

Obviously, if β is large enough we have

$$\frac{(\theta_1^2-3)^{(A-A(\tilde{\sigma}_n))}}{\theta_1^{A-A(\tilde{\sigma}_n)+B-B(\tilde{\sigma}_n)}} \sim \frac{\theta_1^{2(A-A(\tilde{\sigma}_n))}}{\theta_1^{A-A(\tilde{\sigma}_n)+B-B(\tilde{\sigma}_n)}} = \frac{\theta_1^{B(\tilde{\sigma}_n)-A(\tilde{\sigma}_n)}}{\theta_1^{B-A}}.$$

If $B(\tilde{\sigma}_n)-A(\tilde{\sigma}_n)\leqslant B-A$ then the last relation implies that $\frac{(\theta_1^2-3)^{(A-A(\tilde{\sigma}_n))}}{\theta_1^{A-A(\tilde{\sigma}_n)+B-B(\tilde{\sigma}_n)}}$ is bounded, and hence from (2.11) we get the required relation.

Now it remains to prove the following

Lemma 2.11. For every $n>0$ and $\sigma_n\in\Omega_{V_n}$ the following inequality holds

$$B(\sigma_n)-A(\sigma_n)\leqslant B-A. \tag{2.12}$$

Proof. Denote $\mathcal{C}(\sigma_n)=\{x\in V_n:\sigma(x)=-1\}$. Maximal connected components of $\mathcal{C}(\sigma_n)$ we will denote by $\mathcal{K}_1(\sigma_n),\dots,\mathcal{K}_m(\sigma_n)$. For a connected subset $\mathcal{K}$ of V_n put

$$\partial\mathcal{K}=\{x\in V_n\setminus\mathcal{K}:\ \langle x,y\rangle \text{ for some } y\in\mathcal{K}\},$$
$$\partial^2\mathcal{K}=\{x\in V_n\setminus\mathcal{K}:\ >x,y< \text{ for some } y\in\mathcal{K}\}.$$

From definition of $A(\sigma_n)$ and $B(\sigma_n)$ we get

$$B(\sigma_n)=B-2\sum_j|\partial\mathcal{K}_j(\sigma_n)|,$$
$$A(\sigma_n)=A-2\sum_j|\partial^2\mathcal{K}_j(\sigma_n)\setminus\cup_{m\neq j}\mathcal{K}_m(\sigma_n)|,$$

here $|A|$ stands for a number of elements of a set A.

To prove (2.12) it enough to show that $|\partial^2\mathcal{K}|\leqslant|\partial\mathcal{K}|$ for all connected subsets $\mathcal{K}$ of V_n. For each $x\in\partial^2\mathcal{K}$ we can show a $y=y(x)\in\partial\mathcal{K}$. Indeed, if $>x,t<,\ \ t\in\mathcal{K}$ and $\langle x,z,t\rangle$ then $y(x)=x$ if $z\in\mathcal{K}$ and $y(x)=z$ if $z\notin\mathcal{K}$. It is clear $y(x)\in\partial\mathcal{K}$. Now we will prove that if $x_1\neq x_2\in\partial^2\mathcal{K}$ then $y(x_1)\neq y(x_2)\in\partial\mathcal{K}$. Let $\langle x_1,z_1,t_1\rangle,\langle x_2,z_2,t_2\rangle$, where $t_1,t_2\in\mathcal{K}$. By definition of $y(x)$ we have $y(x_i)\in\{x_i,z_i\}, i=1,2$. So to prove $y(x_1)\neq y(x_2)$ for $x_1\neq x_2$ it is enough to show $z_1\neq z_2$. Note that in our case (i.e. $k=2$) if $x_1\neq x_2\in\partial^2\mathcal{K}$

then $d(x_1, x_2) \geqslant 3$, since $\langle x_i, z_i \rangle, i = 1, 2$ and hence we get $z_1 \neq z_2$. Thus $|\partial^2 \mathcal{K}| \leqslant |\partial \mathcal{K}|$. So Lemma is proved.

By similar argument the theorem can be proved for the measure μ_1. Thus the theorem is proved. ▮

From Theorem 2.10 we conclude that σ_+ and σ_- are ground states of the considered model.

Remark 2.5. If in the condition of Theorem 2.10, we put $J = 0$ then the obtained result coincides with Theorem 2.3 of ref. 3.

Now introduce two configurations as follows

$$\sigma_{+-} = \{\sigma_{+-}(x),\ x \in V\}, \quad \sigma_{-+} = \{\sigma_{-+}(x),\ x \in V\},$$

where

$$\sigma_{+-}(x) = \begin{cases} 1, & \text{if } x \in G_2^{(2)}, \\ -1, & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \qquad \sigma_{-+}(x) = \begin{cases} -1, & \text{if } x \in G_2^{(2)}, \\ 1, & \text{if } x \in G_2 \setminus G_2^{(2)}. \end{cases}$$

We can formulate the following

Theorem 2.12. Let $\theta_1 < 1/\sqrt{3}$ and $\theta > \dfrac{2\theta_1}{1 - 3\theta_1^2}$ then

$$\mu_{12} \to \delta_{\sigma_{-+}}, \quad \mu_{21} \to \delta_{\sigma_{+-}} \quad \text{as} \quad \beta \to \infty.$$

The proof is similar to the proof of Theorem 2.10.

3. DIAGONAL STATES GENERATED BY GIBBS MEASURES AND CORRESPONDING VON NEUMANN ALGEBRAS

In this section we consider a case $\theta_1 > \sqrt{3}, \theta > \dfrac{2\theta_1}{\theta_1^2 - 3}$ and determine types of von Neumann algebras generated by the GNS-representation associated with the diagonal states corresponding to the translation invariant measures.

As the paper[11] we consider C^*-algebra $A = \otimes_{\Gamma^k} M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the algebra of 2×2 matrices over the field $\mathbb{C}$ of complex numbers.

By ω_i $(i = 1, 2, 3)$ we denote the diagonal state generated by the translation invariant measures μ_1, μ_2, μ_3 respectively. On the finite dimensional C^*-subalgebra $A_{V_n} = \otimes_{V_n} M_2(\mathbb{C}) \subset A$ we rewrite the state ω_i as follows

$$\omega_i(x) = \frac{tr(e^{\tilde{H}_i(V_n)} x)}{tr(e^{\tilde{H}_i(V_n)})}, \quad x \in A_{V_n}, \tag{3.1}$$

where tr is a trace on A_{V_n}. The term $\sigma(x)\sigma(y)$, $(>x,y<)$ in (2.2) we represent as a diagonal element of $M_2(\mathbb{C})\otimes M_2(\mathbb{C})\otimes M_2(\mathbb{C})$ in the standard basis as follows

$$\sigma(x)\sigma(y)=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}. \tag{3.2}$$

Using (3.2), the form of Hamiltonian (1.1),(2.1) and (3.1) the Hamiltonian $\tilde{H}_i(V_n)$ in the standard basis of A_{V_n} has the form

$$\tilde{H}_i(V_n)=\sum_{>x,y<:x,y\in V_n} F_{>x,y<}+\sum_{\langle x,y\rangle:x,y\in V_n} G_{\langle x,y\rangle}+\sum_{x\in W_n} h_i\sigma_x^z,$$

here and below

$$F_{>x,y<}=\begin{pmatrix} A\otimes\mathbb{1} & O\\ O & B\otimes\mathbb{1}\end{pmatrix},\quad A=\begin{pmatrix}\log p_1 & 0\\ 0 & \log p_2\end{pmatrix},\quad B=UAU,\quad U=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{3.3}$$

$$p_1=\frac{1}{e^{-2\beta J}+1},\quad p_2=1-p_1=\frac{e^{-2bJ}}{e^{-2\beta J}+1} \tag{3.4}$$

$$G_{\langle x,y\rangle}=\begin{pmatrix} A_1 & O\\ O & UA_1U\end{pmatrix},\quad A_1=\begin{pmatrix}\log p_{11} & 0\\ 0 & \log p_{22}\end{pmatrix}, \tag{3.5}$$

$$p_{11}=\frac{1}{e^{-2\beta J_1}+1},\quad p_{22}=1-p_{11}=\frac{e^{-2bJ_1}}{e^{-2\beta J_1}+1}\quad \sigma_x^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{3.6}$$

and $h_i=\log u_i^*/2$, where u_i^* is a solution of (2.5).

Hence the state ω_i is an Gibbs state for quantized Hamiltonian

$$\tilde{H}=\sum_{>x,y<} F_{>x,y<}+\sum_{\langle x,y\rangle} G_{\langle x,y\rangle}.$$

Denote $\mathcal{M}_i=\pi_{\omega_i}(A)''$, where $\pi_{\omega_i}-$ is a GNS representation associated with ω_i (see ref. [4, definition 2.3.18]). Note $\mathcal{M}_i$ is a factor, since the measures μ_i $(i=1,2,3)$ are translation invariant and satisfy mixing property, i.e.

$$\lim_{|g|\to\infty} \omega_i(T_g(x)y) = \omega_i(x)\omega_i(y),$$

here T_g is a left shift transformation of G_2. Our goal in the present section is to determine a type of $\mathcal{M}_i$.

We note that the modular group of $\mathcal{M}_i$ associated with ω_i is defined by

$$\sigma_t^{\omega_i}(x) = \lim_{V_n\to V} \exp\{it\tilde{H}_i(V_n)\}x\exp\{-it\tilde{H}_i(V_n)\}, \quad x\in\mathcal{M}_i. \tag{3.7}$$

here as before

$$\tilde{H}(\Lambda) = \sum_{>x,y<:x,y\in V_n} F_{>x,y<} + \sum_{\langle x,y\rangle:x,y\in V_n} G_{\langle x,y\rangle} + \sum_{x\in W_n} h_i\sigma_x^z.$$

The existence of the last limit easily can be checked by using Theorem 6.2.4[(5)] (see ref. 11).

Lemma 3.1. Let the following condition be satisfied: there exist integers k_i and $m_j^{(i)}$, $j\in\{1,2,3\}$ and the smallest number $\delta_i\in(0,1)$ such that

$$\frac{p_1}{p_2} = \delta_i^{m_1^{(i)}}, \quad \frac{p_{11}}{p_{22}} = \delta_i^{m_2^{(i)}}, \quad \frac{p_1}{p_{11}} = \delta_i^{m_3^{(i)}}, \quad \exp\{h_i\} = \delta_i^{k_i}, \tag{3.8}$$

then for the modular group $\sigma_t^{\omega_i}$ and the number $t_0 = -2\pi/\log\delta_i$, the equality holds

$$\sigma_{t_0}^{\omega_i} = Id.$$

Proof. From (3.8) we have

$$\left.\begin{array}{ll} p_1 = \dfrac{\delta_i^{m_1^{(i)}}}{\delta^{m_1^{(i)}}+1}, & p_2 = \dfrac{1}{\delta_i^{m_1^{(i)}}+1}, \\ p_{11} = \dfrac{\delta_i^{m_1^{(i)}-m_3^{(i)}}}{\delta_i^{m_1^{(i)}}+1}, & p_{22} = \dfrac{\delta_i^{m_1^{(i)}-m_2^{(i)}-m_3^{(i)}}}{\delta_i^{m_1^{(i)}}+1}. \end{array}\right\} \tag{3.9}$$

Hence from (3.3), (3.4) and (3.9) we can get that $\sigma_{t_0}^{\omega_i} = Id$. This completes the proof. ▮

Now using Lemma 3.1, Proposition 5.2 ref. 11 and the argument of ref. 11 we can prove the following

Theorem 3.2. Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the condition (3.8) be satisfied. Then von Neumann algebras $\mathcal{M}_i$ corresponding to the translation invariant Gibbs states μ_i of the Ising model with competing interactions (1.1) on the Cayley tree Γ^2 are factors of type III_{δ_i}.

Since u_1^* and u_3^* are the solution of the equation (2.6) then from (3.8) we find that $k_1 = -k_3$ and $\delta_1 = \delta_3$. This implies that the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ have the same type. It is easy to see that $k_2 = 0$.

From Theorem 3.2 and the argument of[11] we have the following

Corollary 3.3. Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the following condition be satisfied: there exist integers $m_i, i \in \{1, 2, 3\}$ and the smallest number $\delta \in (0, 1)$ such that

$$\frac{p_1}{p_2} = \delta^{m_1}, \quad \frac{p_{11}}{p_{22}} = \delta^{m_2}, \quad \frac{p_1}{p_{11}} = \delta^{m_3}, \tag{3.10}$$

then a von Neumann algebras $\mathcal{M}_2$ corresponding to the unordered phase of the Ising model with competing interactions (1.1) on the Cayley tree Γ^2 is a factor of type III_δ. Otherwise $\mathcal{M}_2$ is a factor of type III_1.

Corollary 3.4. Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and the following conditions be satisfied: there exist integers k and $n_i, i \in \{1, 2, 3\}$ and the smallest number $\delta_1 \in (0, 1)$ such that

$$\frac{p_1}{p_2} = \delta_1^{n_1}, \quad \frac{p_{11}}{p_{22}} = \delta_1^{n_2}, \quad \frac{p_1}{p_{11}} = \delta_1^{n_3}, \tag{3.11}$$

and

$$\exp\{h_1\} = \delta_1^k, \quad h_1 > 0, \tag{3.12}$$

then von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_3$ corresponding to the Gibbs states μ_1 and μ_3 respectively, of the Ising model with competing interactions (1.1) on the Cayley tree Γ^2 are factors of type III_{δ_1}. Otherwise they are factors of type III_1.

Remark 3.1. If we consider the case $\theta_1 < 1/\sqrt{3}$ and $\theta > \frac{2\theta_1}{1-3\theta_1^2}$, then there are two strictly periodic (non translation invariant) Gibbs measures. By similar arguments as above we can prove analogical theorems as Theorem 3.2 for these periodic measures.

Remark 3.2. Here it would be good to mention that there is an example of factor generated by Cayley tree, but it does not appear from a physical system (see ref. 12).

It is clear that if (3.10) is not satisfied then (3.11) is too, consequently, the algebras $\mathcal{M}_i$ are factors of type III_1. Suppose (3.10) is valid then (3.11) is also satisfied with $\delta_1 \geqslant \delta$, more exactly, $\delta_1 = \delta^r$, where $r \in (0, 1] \cap \mathbb{Q}$. But it is interesting whether the equality $\delta_1 = \delta$ is satisfied. The following theorem answers to this question.

Theorem 3.5. Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ be satisfied. Suppose the equalities (3.10)–(3.12) are satisfied. Then the factor $\mathcal{M}_1$ and $\mathcal{M}_3$ have types $\mathrm{III}_{\delta^r}, 0 < r < 1, r \in \mathbb{Q}$, while the factor $\mathcal{M}_2$ has type III_δ.

Proof. The conditions (3.10) and (3.12) imply that there exists a rational number $s \in \mathbb{Q}$ such that $\theta = \theta_1^s$. We note this is a necessary condition that M_2 to be a factor of type III_δ. We want to prove that $\delta_1 > \delta$. Let us assume that $\delta_1 = \delta$. Keeping in mind that the numbers e^{2h_1} and e^{-2h_1} are the solutions of the Eq. (2.6) from (3.12) we obtain

$$2\cosh(2k\log\delta) = \theta_1^2 - 2\theta_1^{1-s} - 1, \tag{3.13}$$

here we have used that $\alpha = 2\theta_1^{1-s} - \theta_1^2$. From (3.11) we find that $\log\delta = -2n_1 J_1 \beta$, substituting it into (3.13) we have

$$2\cosh(4nJ_1\beta) = \theta_1^2 - 2\theta_1^{1-s} - 1, \tag{3.14}$$

here without loss of generality we may assume that $n > 0$, $n \in \mathbb{Z}$, since $\cosh(x)$ is an even function. Defining $f(n) = 2\cosh(4nJ_1\beta)$ from (3.14) it is easy to see that $f(1) > \theta_1^2 - 2\theta_1^{1-s} - 1$, since $\theta_1 > \sqrt{3}$. It is clear that $f(n)$ is an increasing function, so this implies that the equality (3.14) can not be satisfied for any positive integer n. This means $\delta_1 > \delta$. Consequently, the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ can not have the same type with the factor $\mathcal{M}_2$. This completes the proof. ∎

The proved Theorem means that the factor $\mathcal{M}_2$ can be considered as a subfactor of $\mathcal{M}_1$ and $\mathcal{M}_3$, respectively.

Corollary 3.6. Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ be satisfied. If there is an irrational γ such that $J = \gamma J_1$ then the factors $\mathcal{M}_i$ $(i = 1, 2, 3)$ have type III_1.

Let us consider some more concrete examples of factors.

Example 3.1. Suppose that $J = 0$ and $\theta_1 > \sqrt{3}$. Then the condition $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ implies that $\theta_1 > 3$. In this case the Eq. (3.10) reduces to the following one

$$\frac{p_{11}}{p_{22}} = \delta^m,$$

here as before $\delta \in (0, 1)$ and $m \in \mathbb{Z}$, which is automatically satisfied with $\delta = \theta_1^{-1}$ and $m = -1$. So in this case $\mathcal{M}_2$ is a factor of type III_δ. But it is interesting question is whether the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ can have type III_{δ_1}, while the factor $\mathcal{M}_2$ has type III_δ. Now we going to show that this can be occur.

Indeed, we firstly note that in the considered case the Eq. (2.4) can be written as follows

$$h = 2\operatorname{arctanh}(\tilde{\theta} \tanh h), \tag{3.15}$$

here $\tilde{\theta} = \tanh(J_1\beta)$ (see, ref. 2). We recall that the condition $\theta_1 > 3$ is equivalent to $\tilde{\theta} > \frac{1}{2}$. Now using the formula

$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x},$$

from (3.15) we obtain

$$\tanh h_1 = \frac{2\tilde{\theta}\tanh h_1}{1 + (\tilde{\theta}\tanh h_1)^2},$$

it yields that

$$h_1 = \operatorname{arctanh}\left(\frac{\sqrt{2\tilde{\theta} - 1}}{\tilde{\theta}}\right). \tag{3.16}$$

Let us turn to the conditions (3.11) and (3.12). In our case they can be reduced to the following ones

$$\frac{p_{11}}{p_{22}}=\delta_1^n, \quad \exp\{h_1\}=\delta_1^k, \quad n,k\in\mathbb{Z}. \tag{3.17}$$

Choose the number $\tilde{\theta}$ such that which satisfies the following equation

$$\tilde{\theta}^3+5\tilde{\theta}^2+7\tilde{\theta}-5=0. \tag{3.18}$$

It is not hard to check that the required $\tilde{\theta}$ does exist, i.e. with the property $1/2<\tilde{\theta}<1$. Put $\delta_1=\sqrt[4]{\delta}$ or $\delta_1=\theta_1^{-1/4}$. It easy to see that for such δ_1 we have $n=-4$. From (3.17) we find

$$h_1=-\frac{k}{2}J_1\beta,$$

which yields

$$h_1=-\frac{k}{2}\text{arctanh}\tilde{\theta}, \tag{3.19}$$

here we have used the $J_1\beta=\text{arctanh}\tilde{\theta}$.

We will show that this equality is satisfied when $k=-1$. Indeed, using (3.16) from (3.19) we have

$$\frac{\sqrt{2\tilde{\theta}-1}}{\tilde{\theta}}=\tanh\left(\frac{\text{arctanh}(\tilde{\theta})}{2}\right). \tag{3.20}$$

Now according to the formula

$$\tanh\frac{x}{2}=\frac{1-\sqrt{1-\tanh^2 x}}{\tanh x}, \quad x>0$$

from (3.20) we get

$$\sqrt{2\tilde{\theta}-1}+\sqrt{1-\tilde{\theta}^2}=1.$$

The last equation equivalent to the following one

$$(\tilde{\theta}-1)(\tilde{\theta}^3+5\tilde{\theta}^2+7\tilde{\theta}-5)=0.$$

The condition (3.18) yields that the last equality is satisfied, hence (3.20) is valid. Thus the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ have type $\text{III}_{\sqrt[4]{\delta}}$.

These results clarifies and specifies the results obtained in refs. 10 and 11.

Example 3.2. Suppose that $J = J_1$ and $J \neq 0$, this means $\theta = \theta_1$. Hence the equality (3.10) is satisfied with parameters: $\delta = \theta^{-1}, m_1 = -1, m_2 = -1, m_3 = 0$. So according to Corollary 3.3 we conclude that $\mathcal{M}_2$ is a factor of type III_δ. Now assume that there is a phase transition, i.e. the condition $\theta_1 > \sqrt{3}, \theta > \dfrac{2\theta_1}{\theta_1^2 - 3}$ is satisfied, which implies in our case (i.e., $J = J_1$) that $\theta > \sqrt{5}$. Now we are going to find another δ_1 for which the factors $\mathcal{M}_1$ and $\mathcal{M}_3$ have type III_{δ_1}.

Put $\theta = 1 + \sqrt{2}$ and $\delta_1 = \sqrt{\delta}$. It is clear that (3.11) is satisfied. Now we should check (3.12). Keeping in mind that the numbers e^{2h_1} and e^{-2h_1} are the solutions of (2.6) from (3.12) we get

$$\delta^k + \delta^{-k} = \theta^2 - 3. \tag{3.21}$$

Put $k = 1$. Let us show this equality is satisfied. The equality (3.21) can be written as follows

$$(\theta + 1)(\theta^2 - 2\theta - 1) = 0.$$

The chosen θ satisfies this equation, hence (3.12) is valid. This is the required.

If $J = 0$ then the phase transition does not occur and the factor $\mathcal{M}_2$ has type II_1. We note in the case $\theta_1 > \sqrt{3}$ and $\theta > \dfrac{2\theta_1}{\theta_1^2 - 3}$ the factor $\mathcal{M}_2$ can not have a type II_1.

4. DISCUSSION OF THE RESULTS

It is known that to exact calculations in statistical mechanics are paid attention by many of researchers, because those are important not only for their own interest but also for some deeper understanding of the critical properties of spin systems which are not obtained form approximations. So, those are very useful for testing the credibility and efficiency of any new method or approximation before it is applied to more complicated spin systems. In the previous paper[11] we have exactly solved an Ising model on a Cayley tree, the Hamiltonian of which contained ternary interactions. In addition, we found some conditions on parameters which enabled to determine exactly types of von Neuamann algebras associated with periodic Gibbs states of that model. In the present paper we continue investigations of the Ising model, but now we consider a model with the next-nearest-neighbor binary interactions. Using the same way as[11] we exactly solve a phase transition problem for the model, namely, we calculated critical curve such that there is a phase transitions above it, and a single Gibbs state is found elsewhere. Comparing with the results of ref. 11 in the present paper we describe all periodic Gibbs states associated with subgroups of G_2 with finite index, while in the mentioned paper we only found $G_2^{(2)}$-periodic Gibbs states. Besides, we also find ground states of the considered model. Here (in the paper) as in ref.11 we also find some conditions of parameters J and J_1 which completely determine types of von Neumann algebras corresponding to the translation-invariant Gibbs states, but now we show how these algebras related with each other, more precisely speaking, we prove that the factor corresponding to the unordered phase is a sub-factor of the factors associated with the minimum and maximum Gibbs states. We note that this kind of question was not considered in ref. 11. Finally, we demonstrate some more concrete examples of such factors, which clarify the results obtained in refs. 10 and 11.

We note that some computer simulations results of the model considered were studied in ref. 9. Some other phase transitions problems were considered in ref. 8.

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